

Graph Parameters and Ramsey Theory

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Abstract

Ramsey's Theorem tells us that there are exactly two minimal hereditary classes containing graphs with arbitrarily many vertices: the class of complete graphs and the class of edgeless graphs. In other words, Ramsey's Theorem characterizes the graph vertex number in terms of minimal hereditary classes where this parameter is unbounded. In the present paper, we show that a similar Ramsey-type characterization is possible for a number of other graph parameters, such as vertex cover number, matching number, neighbourhood diversity, VC-dimension.

1 Introduction

In 1930, a 26 years old British mathematician Frank Ramsey proved the following theorem, known nowadays as Ramsey's Theorem.

Theorem 1. [13] *For any positive integers k, r, p , there exists a minimum positive integer $F = F(k, r, p)$ such that if the k -subsets of an F -set are colored with r colors, then there is a monochromatic p -set, i.e. a p -set all of whose k -subsets have the same color.*

We will refer to the number $F(k, r, p)$ defined in this theorem as the *Frank Ramsey number*. It is not difficult to see that with $k = 1$ the theorem coincides with the Pigeonhole Principle.

For $k = 2$, the theorem admits a nice interpretation in the terminology of graph theory, since coloring 2-subsets can be viewed as coloring the edges of a complete graph. In the case of $r = 2$ colors, the graph-theoretic interpretation of Ramsey's Theorem can be further rephrased as follows.

Theorem 2. *For any positive integer p , there is a minimum positive integer $R = R(p)$ such that every graph with at least R vertices has either a clique of size p or an independent set of size p .*

The number $R(p) = F(2, 2, p)$ defined in this theorem is known as the *symmetric Ramsey number*. By $R(p, q)$ we denote the non-symmetric version of Ramsey number, i.e. the minimum number of vertices that guarantees the presence of a clique of size p or an independent set of size q . It is not difficult to see that Theorem 2 is equivalent to the following statement.

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Theorem 3. *The class of complete graphs and the class of edgeless graphs are the only two minimal infinite hereditary classes of graphs.*

Theorem 3 characterizes the family of hereditary classes containing graphs with a bounded number of vertices in terms of minimal “forbidden” elements, i.e. minimal classes where the vertex number is unbounded. In the subsequent sections, we provide a similar characterization for several other graph parameters, such as vertex cover number, matching number, neighbourhood diversity, VC-dimension. In the rest of the present section, we introduce basic definitions and notations used in the paper.

We consider only simple undirected graphs without loops and multiple edges and denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. If v is a vertex of G , then $N(v)$ is its neighbourhood, i.e. the set of vertices of G adjacent to v . The closed neighbourhood of v is defined and is denoted as $N[v] = N(v) \cup \{v\}$. The degree of v is $|N(v)|$.

For a graph G , we denote by \overline{G} the complement of G . Similarly, for a class \mathcal{X} of graphs, we denote by $\overline{\mathcal{X}}$ the class of complements of graphs in \mathcal{X} .

Given a graph G and a subset $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U , i.e. the subgraph obtained from G by deleting all the vertices not in U . We say that a graph G contains a graph H as an induced subgraph if H is isomorphic to an induced subgraph of G . A graph G is said to be n -universal for a class of graphs \mathcal{X} if G contains all n -vertex graphs from \mathcal{X} as induced subgraphs.

A class \mathcal{X} of graphs is *hereditary* if it is closed under taking induced subgraphs, i.e. if $G \in \mathcal{X}$ implies $H \in \mathcal{X}$ for every graph H contained in G as an induced subgraph. Two hereditary classes of particular interest in this paper are split graphs and bipartite graphs.

A graph G is a *split* graph if $V(G)$ can be partitioned into an independent set and a clique, and G is *bipartite* if $V(G)$ can be partitioned into at most two independent sets. A bipartite graph G given together with a bipartition of its vertices into independent sets A and B will be denoted $G = (A, B, E)$, in which case we will say that A and B are the color classes of G . If every vertex of A is adjacent to every vertex of B , then $G = (A, B, E)$ is *complete bipartite*. The *bipartite complement* of $G = (A, B, E)$ is the bipartite graph $G' = (A, B, E')$, where $ab \in E'$ if and only if $ab \notin E$. Clearly, by creating a clique in one of the color classes of a bipartite graph, we transform it into a split graph, and vice versa.

2 Matching number and vertex cover number

In a graph, a *matching* is a subset of edges no two of which share a vertex. The *matching number* of a graph G is the size of a maximum matching in G and we denote it by $\mu(G)$. A *vertex cover* is a subset of vertices covering all the edges of the graph. The *vertex cover number* of a graph G is the size of a minimum vertex cover in G and we denote it by $\tau(G)$.

It is well known (and not difficult to see) that $\mu(G) \leq \tau(G) \leq 2\mu(G)$. Therefore, $\mu(G)$ is bounded in a hereditary class if and only if $\tau(G)$ is bounded. To provide a Ramsey-type characterization for both parameters, we denote by

\mathcal{B} the class of complete bipartite graphs (an edgeless graph is counted as complete bipartite with one part being empty). A complete bipartite graph with n vertices in each part of its bipartition will be denoted by B_n . Clearly, B_n is n -universal for graphs in \mathcal{B} .

\mathcal{M} the class of graphs of vertex degree at most 1. By M_n we denote an induced matching of size n , i.e. the unique up to isomorphism graph in the class \mathcal{M} with $2n$ vertices each of which has degree 1. Clearly, M_n is n -universal for graphs in \mathcal{M} .

\mathcal{Z} the class of *chain graphs*, i.e. bipartite graphs in which the neighbourhoods of the vertices in each part form a chain with respect to set-inclusion. By Z_n we denote a chain graph such that for each $i \in \{1, 2, \dots, n\}$, each part of the graph contains exactly one vertex of degree i . Figure 1 represents the graph Z_n for $n = 5$. It is known [9] that Z_n is n -universal for graphs in \mathcal{Z} .

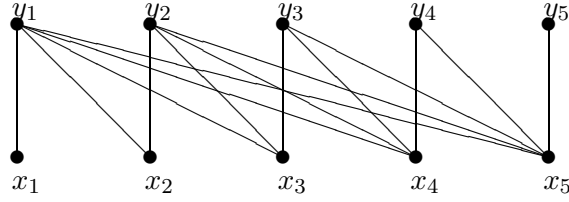


Figure 1: The graph Z_5

Lemma 1. *For any positive integers s, t , there exists a positive integer $q = q(s, t)$ such that every bipartite graph G with a matching of size q contains either an induced M_s or an induced B_t .*

Proof. Let us denote $m = 2 \max(s, t)$ and $q = F(2, 4, m)$, where $F(k, r, p)$ is the Frank Ramsey number. Consider a matching $M = \{x_1y_1, \dots, x_qy_q\}$ of size q . We color each pair (x_iy_i, x_jy_j) of edges in M ($i < j$) in one of the four colors as follows:

- color 1 if G contains no edges between x_iy_i and x_jy_j ,
- color 2 if G contains both edges between x_iy_i and x_jy_j ,
- color 3 if G contains the edge x_iy_j but not the edge y_ix_j ,
- color 4 if G contains the edge y_ix_j but not the edge x_iy_j .

By Ramsey's Theorem, M contains a monochromatic set M' of edges of size m . If the color of each pair in M' is

- 1, then M' is an induced matching of size $m \geq 2s > s$,
- 2, then the vertices of M' induce a complete bipartite graph B_m with $m \geq 2t > t$,
- 3 or 4, then the vertices of M' induce a Z_m and hence G contains a complete bipartite graph $B_{m/2}$ with $m/2 \geq t$.

□

Lemma 2. *For any natural s, t, p , there exists a $Q = Q(s, t, p)$ such that every graph G with a matching of size Q contains either an induced M_s or an induced B_t or a clique K_p .*

Proof. Let $Q = R(p, R(p, q))$, where R is the (non-symmetric) Ramsey number and $q = F(2, 4, 2 \max(s, t))$ is the value defined in the proof of Lemma 1. We consider a matching M of size Q in G and color the endpoints of each edge of M in two colors, say white and black, arbitrarily. Since the set of white vertices has size Q , it must contain either a clique K_p , in which case we are done, or an independent set A of size $R(p, q)$. In the latter case, we look at the black vertices matched with the vertices of A . According to the size of this set, it must contain either a clique K_p , in which case we are done, or an independent set A' of size q . In the latter case, we denote by A'' the set of white vertices matched with the vertices of A' . Then A' and A'' induce a bipartite graph with a matching of size q , in which case, by Lemma 1, G contains either an induced matching of size s or an induced complete bipartite graph B_t . \square

Lemma 2 allows us to make the following conclusion.

Theorem 4. *\mathcal{M} , \mathcal{B} and the class of complete graphs are the only three minimal hereditary classes of graphs of unbounded matching number and unbounded vertex cover number.*

3 Neighbourhood diversity

The neighbourhood diversity of a graph was introduced in [7] and can be defined as follows. Let us say that two vertices x and y are *similar* if there is no vertex z distinguishing them (i.e. if there is no vertex z adjacent to exactly one of x and y). Clearly, the similarity is an equivalence relation. The *neighbourhood diversity* of G is the number of similarity class in G .

In order to characterize the neighbourhood diversity by means of minimal hereditary classes of graphs where this parameter is unbounded, we denote by

\mathcal{M}^{bc} the class of bipartite complements of graphs in \mathcal{M} . The bipartite complement of the graph M_n will be denoted M_n^{bc} . Clearly, M_n^{bc} is n -universal for graphs in \mathcal{M}^{bc} .

\mathcal{M}^* the class of split graphs obtained from graphs in \mathcal{M} by creating a clique in one of the color classes. The graph obtained from M_n by creating clique in one its color classes will be denoted by M_n^* . Clearly, M_n^* is n -universal for graphs in \mathcal{M}^* .

\mathcal{Z}^* the class of split graphs obtained from graphs in \mathcal{Z} by creating a clique in one of the color classes. This class is known in the literature as the class of *threshold graphs*. The graph obtained from Z_n by creating a clique in one of its color classes will be denoted Z_n^* . This graph is n -universal for threshold graphs [5].

Before we provide a characterization of the neighbourhood diversity, we introduce an auxiliary parameter.

Definition 1. A *skew matching* in a graph G is a matching $\{x_1y_1, \dots, x_qy_q\}$ such that y_i is not adjacent to x_j for all $i < j$. The *complement of a skew matching* is a sequence of pairs of vertices that create a skew matching in the complement of G .

Lemma 3. *For any positive integer m , there exists a positive integer $r = r(m)$ such that any bipartite graph $G = (A, B, E)$ of neighbourhood diversity r contains either a skew matching of size m or its complement.*

Proof. Define $r = 2^{2m}$ and let X be a set of pairwise non-similar vertices of size $r/2$ chosen from the same color class of G , say from A . Let y_1 be a vertex in B distinguishing the set X (i.e. y_1 has both a neighbour and a non-neighbour in X) and let us say that y_1 is *big* if the number of its neighbours in X is larger than the number of its non-neighbours, and *small* otherwise. If y_1 is small, we arbitrarily choose its neighbour in X , denote it by x_1 and remove all neighbours of y_1 from X . If y_1 is big, we arbitrarily choose a non-neighbour of y_1 in X , denote it by x_1 and remove all non-neighbours of y_1 from X . Observe that y_1 does not distinguish the vertices in the updated set X .

We apply the above procedure to X $2m - 1$ times and obtain in this way a sequence of $2m - 1$ pairs $x_i y_i$. If m of these pairs contain small vertices y_i , then the respective pairs create a skew matching of size m . Otherwise, there is a set of m pairs containing big vertices y_i , in which case the respective pairs create the complement of a skew matching. \square

Lemma 4. *For any positive integer p , there exists a positive integer $q = q(p)$ such that any bipartite graph $G = (A, B, E)$ of neighbourhood diversity q contains either an induced M_p or an induced Z_p or an induced M_p^{bc} .*

Proof. Let $m = R(p + 1)$ (where R is the symmetric Ramsey number) and $q = 2^{2m}$. According to the proof of Lemma 3, G contains a skew matching of size m or its complement. If G contains a skew matching M , we color each pair $(x_i y_i, x_j y_j)$ of edges of M ($i < j$) in two colors as follows:

- color 1 if x_i is not adjacent to y_j ,
- color 2 if x_i is adjacent to y_j .

By Ramsey's Theorem, M contains a monochromatic set M' of edges of size $p + 1$. If the color of each pair of edges in M' is

- 1, then M' is an induced matching of size $p + 1$,
- 2, then the vertices of M' induce a Z_{p+1} .

Analogously, in the case when G contains the complement of a skew matching, we find either an induced M_{p+1}^{bc} or an induced Z_p (observe that the bipartite complement of Z_{p+1} contains an induced Z_p). \square

Lemma 5. *For any positive integer p , there exists a positive integer $Q = Q(p)$ such that every graph G of neighbourhood diversity Q contains one of the following nine graphs as an induced subgraph: M_p , M_p^{bc} , Z_p , \overline{M}_p , \overline{M}_p^{bc} , \overline{Z}_p , M_p^* , \overline{M}_p^* , Z_p^* .*

Proof. Let $Q = R(q)$, where $q = 2^{2m}$ and $m = R(R(p) + 1)$ (R is the symmetric Ramsey number). We choose one vertex from each similarity class of G and find in the chosen set a subset A of vertices that form an independent set or a clique of size $q = 2^{2m}$. Let us call the vertices of A white. We denote the remaining vertices of G by B and call them

black. Let G' denote the bipartite subgraph of G formed by the edges between A and B . By the choice of A , all vertices of this set have pairwise different neighbourhoods in G' . Therefore, according to the proof of Lemma 4, G' contains a subgraph G'' inducing either M_n , or M_n^{bc} or Z_n with $n = R(p)$. Among the n black vertices of G'' , we can find a subset B' of vertices that form either a clique or an independent set of size p in the graph G . Then B' together with a subset of A of size p induce in G one of the nine graphs listed in the statement of the theorem. \square

Since the nine graphs of Lemma 5 are universal for their respective classes, we make the following conclusion.

Theorem 5. *There exist exactly nine minimal classes of graphs of unbounded neighbourhood diversity: \mathcal{M} , \mathcal{M}^{bc} , \mathcal{Z} , $\overline{\mathcal{M}}$, $\overline{\mathcal{M}}^{bc}$, $\overline{\mathcal{Z}}$, \mathcal{M}^* , $\overline{\mathcal{M}}^*$, \mathcal{Z}^* .*

4 VC-dimension

A set system (X, S) consists of a set X and a family S of subsets of X . A subset $A \subseteq X$ is *shattered* if for every subset $B \subseteq A$ there is a set $C \in S$ such that $B = A \cap C$. The VC-dimension of (X, S) is the cardinality of a largest shattered subset of X .

The VC-dimension of a graph $G = (V, E)$ was defined in [2] as the VC-dimension of the set system (V, S) , where S the family of closed neighbourhoods of vertices of G , i.e. $S = \{N[v] : v \in V(G)\}$. Let us denote the VC-dimension of G by $vc[G]$.

In this section, we characterize VC-dimension by means of three minimal hereditary classes where this parameter is unbounded. To this end, we first redefine it in terms of open neighbourhoods as follows. Let $vc(G)$ be the size of a largest set A of vertices of G such that for any subset $B \subseteq A$ there is a vertex v *outside of* A with $B = A \cap N(v)$. In other words, $vc(G)$ is the size of a largest subset of vertices shattered by *open* neighbourhoods of vertices of G .

We start by showing that the two definitions are equivalent in the sense that they both are either bounded or unbounded in a hereditary class. To prove this, we introduce the following terminology. Let A be a set of vertices which is shattered by a collection of closed neighbourhoods. For a subset $B \subseteq A$ we will denote by $v(B)$ the vertex whose neighbourhood intersect A at B . We will say that B is *closed* if $v(B)$ belongs to B , and *open* otherwise.

Lemma 6. $vc(G) \leq vc[G] \leq vc(G)(vc(G) + 1) + 1$

Proof. The first inequality is obvious. To prove the second one, let A be a subset of $V(G)$ of size $vc[G]$ which is shattered by a collection of closed neighbourhoods. If A has no closed subsets, then $vc[G] = vc(G)$. Otherwise, let B be a closed subset of A .

Assume first that $|B| = 1$. Then $B = \{v(B)\}$ and $v(B)$ is isolated in $G[A]$, i.e. it has no neighbours in A . Let C be the set of all such vertices, i.e. vertices each of which is a closed subset of A . By removing from A any vertex $x \in C$ we obtain a new set A and may assume that it has no closed subsets of size 1. Indeed, for any vertex $y \in C$ different from x , there must exist a vertex $y' \notin A$ such that $N(y') \cap A = \{x, y\}$ (since A is shattered). After the removal of x from A , we have $N(y') \cap A = \{y\}$ and hence $\{y\}$ is not a closed subset anymore. This discussion allows us to assume in what follows that A has no closed subsets of size 1, in which case we only need to show that $vc[G] \leq vc(G)(vc(G) + 1)$.

Assume now that B is a closed subset of A of size at least 2. Suppose that $B - v(B)$ contains a closed subset C , i.e. $v(C) \in C$. Observe that $v(C)$ is adjacent to $v(B)$, as every vertex of $B - v(B)$ is adjacent to $v(B)$. But then $N[v(C)] \cap A$ contains $v(B)$ contradicting the fact that $N[v(C)] \cap A = C$. This contradiction shows that every subset of $B - v(B)$ is open, i.e. $|B - v(B)| \leq vc(G)$.

The above observation allows us to apply the following procedure: as long as A contains a closed subset B with at least two vertices, delete from A all vertices of B except for $v(B)$. Denote the resulting set by A^* . Assume the procedure was applied p times and let B_1, \dots, B_p be the closed subsets it was applied to. It is not difficult to see that the set $\{v(B_1), \dots, v(B_p)\}$ has no closed subsets and hence its size cannot be large than $vc(G)$, i.e. $p \leq vc(G)$. Combining, we conclude:

$$vc[G] = |A| \leq |A^*| + \sum_{i=1}^p |B_i - v(B_i)| \leq vc(G) + p \cdot vc(G) \leq vc(G)(vc(G) + 1).$$

□

This lemma allows us to assume that if A is shattered, then there is a set C *disjoint* from A such that for any subset $B \subseteq A$ there is a vertex $v \in C$ with $B = A \cap N(v)$, in which case we will say that A is shattered by C , or C shatters A .

Let $Q_n = (A, B, E)$ be the bipartite graph with $|A| = n$ and $|B| = 2^n$ such that all vertices of B have pairwise different neighbourhood in A . Also, let S_n be the split graph obtained from Q_n by creating a clique in A .

Lemma 7. *The graph Q_n is an n -universal bipartite graph, i.e. it contains every bipartite graph with n vertices as an induced subgraph.*

Proof. Let G be a bipartite graph with n vertices and with parts A and B of size n_1 and n_2 , respectively. By adding at most n_2 vertices to A , we can guarantee that all vertices of B have pairwise different neighbourhoods in A . Clearly, Q_n contains the extended graph and hence it also contains G as an induced subgraph. □

Corollary 1. *Every co-bipartite graph with at most n vertices is contained in \overline{Q}_n and every split graph with at most n vertices is contained both in S_n and in \overline{S}_n .*

Lemma 8. *If a set A shatters a set B with $|B| = 2^n$, then B shatters a subset A^* of A with $|A^*| = n$.*

Proof. Without loss of generality we assume that B is the set of all binary sequences of length n . Then every vertex $a \in A$ defines a Boolean function of n variables (the neighbourhood of a consists of the binary sequences, where the function takes value 1). For each $i = 1, \dots, n$, let us denote by a_i the Boolean function such that $a_i(x_1, \dots, x_n) = 1$ if and only if $x_i = 1$. Let A' be an arbitrary subset of $A^* = \{a_1, \dots, a_n\}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ its characteristic vector, i.e. $\alpha_i = 1$ if and only if $a_i \in A'$. Clearly, $\alpha \in B$ and $N(\alpha) \cap A^* = A'$. Therefore, B shatters A^* . □

Lemma 9. *For every n , there exists a $k = k(n)$ such that every graph G with $vc(G) = k$ contains one of $Q_n, \overline{Q}_n, S_n, \overline{S}_n$ as an induced subgraph.*

Proof. Define $k = R(2^{R(n)})$, where R is the symmetric Ramsey number. Since $vc(G) = k$, there are two subsets A and B of $V(G)$ such that $|A| = k$ and B shatters A . By definition of k , A must have a subset A' of size $2^{R(n)}$ which is a clique or an independent set. Clearly, B shatters A' and hence, by Lemma 8, A' shatters a subset B' of B of size $R(n)$. Then B' must have a subset B'' of size n which is either a clique or an independent set. Now $G[A' \cup B'']$ is either bipartite or co-bipartite or split graph, $|B''| = n$ and A' shatters B'' . Therefore, $G[A' \cup B'']$ contains one of $Q_n, \overline{Q}_n, S_n, \overline{S}_n$ as an induced subgraph. \square

Theorem 6. *The classes of bipartite, co-bipartite and split graphs are the only three minimal hereditary classes of graphs of unbounded VC-dimension.*

Proof. Clearly these three classes have unbounded VC-dimension, since they contain $Q_n, \overline{Q}_n, S_n, \overline{S}_n$ with arbitrarily large values of n .

Now let X be a hereditary class containing none of these three classes. Therefore, there is a bipartite graph G_1 , a co-bipartite graph G_2 and a split graph G_3 which are forbidden for X . Denote by n the maximum number of vertices in these graphs.

Assume that VC-dimension is not bounded for graphs in X and let $G \in X$ be a graph with $vc(G) = k$, where $k = k(n)$ is from Lemma 9. Then G contains one of $Q_n, \overline{Q}_n, S_n, \overline{S}_n$, say Q_n . Since Q_n is n -universal for bipartite graphs (Lemma 7), it contains G_1 as an induced subgraph, which is impossible because G_1 is forbidden for graphs in X . This contradiction shows that VC-dimension is bounded in the class X . \square

5 Concluding remarks and open problems

The world of graph parameters is rich and diverse and many of them admit a Ramsey-type characterization in terms of minimal hereditary classes, where these parameters are unbounded. In addition to the results presented in this paper, we can mention some other Ramsey-type characterizations.

5.1 What else?

A trivial consequence of Ramsey's Theorem is that there are exactly two minimal hereditary classes of graphs of unbounded vertex degree: the class of complete graphs and the class of stars (and all their induced subgraphs).

A less trivial result of this type was recently obtained in [3]. It states that for every t, p, s , there exists a $z = z(t, p, s)$ such that every graph with a (not necessarily induced) path of length at least z contains either an induced path of length at least t or an induced complete bipartite graph B_p or a clique K_s . Therefore, the class of linear forests (i.e. graphs every connected component of which is a path), the class of complete bipartite graphs and the class of complete graphs are the only three minimal hereditary classes of unbounded path number (the length of a longest path). This result was used in [3] to obtain fpt-algorithms in special classes of graphs for the k -Biclique problem, which is generally W[1]-hard [8].

5.2 What is next?

Most parameters of theoretical or practical importance, such as tree- or clique-width, lie between neighbourhood diversity and VC-dimension. However, not all of them admit a

Ramsey-type characterization in terms of minimal classes (we briefly discuss this issue in the next section). Among the parameters for which such a characterization is possible we can distinguish graph lettericity, which was introduced in [12]. This parameter lies between neighbourhood diversity and linear clique-width in the sense that bounded neighbourhood diversity implies bounded lettericity, while bounded lettericity implies bounded linear clique-width.

The importance of this parameter is also due to the fact that there is an intriguing relationship between graph lettericity and geometric grid classes of permutations introduced recently in [1] (see more on this topic in [6]).

5.3 On the non-existence of minimal classes

Trivially, there exist minimal hereditary classes of unbounded tree-width, such as complete graphs or complete bipartite graphs. The existence of minimal hereditary classes of unbounded clique-width is less trivial and the first two such classes (bipartite permutation and unit interval graphs) have been recently discovered in [11]. However, identifying minimal classes is possible not always. For instance, it is known that tree-width and clique-width are unbounded in the class of graphs of girth (the length of a shortest cycle) at most k for all values of k . However, the limit class of this sequence, with k tending to infinity, consists of all forests, in which case both the tree- and clique-width are bounded. This example shows that there are areas in the universe of hereditary classes, where minimal classes do not exist.

There are several ways to overcome this difficulty. One of them is to reduce the universe. For instance, by reducing the universe of hereditary classes to minor-closed classes of graphs, we conclude that the class of planar graphs is the unique minimal minor-closed class of unbounded tree-width [14] and hence it is the unique minimal minor-closed class of unbounded clique-width.

One more way to overcome the difficulty of non-existence of minimal classes is to employ the notion of boundary classes, which is a relaxation of the notion of minimal classes (see e.g. [10]).

5.4 An open problem

Among the myriad of open problems related to the topic of the paper, we would like to distinguish just one. This problem deals with the tools that can be used to identify minimal classes. All the results in the present paper have been derived with the help of Ramsey's Theorem. A more general result is known as the Canonical Ramsey Theorem [4]. It deals with arbitrarily many colors and in the case of coloring 2-subsets, the result can be stated as follows: for every positive integer ℓ , there exists a positive integer $n = n(\ell)$ such that if the edges of an n -vertex complete graph are colored with arbitrarily many colors, then the graph contains a clique of size ℓ which is either monochromatic (all edges have the same color) or rainbow (all edges have pairwise different colors) or skew (the vertices of the clique can be ordered x_1, x_2, \dots, x_ℓ in such a way that the edges $x_i x_j$ ($i < j$) and $x_p x_t$ ($p < t$) have the same color if and only if $i = p$). In this statement, monochromatic and rainbow colorings are complement to each other, while a skew coloring is self-complementary. This resembles the three minimal hereditary classes of bipartite graphs of unbounded neighbourhood diversity: the classes \mathcal{M} and \mathcal{M}^{bc} are

complement to each other, while the class \mathcal{Z} of chain graphs is self-complementary (in the bipartite sense). We ask whether Lemma 4 can be derived directly from the Canonical Ramsey Theorem, avoiding intermediate steps.

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